JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **223,** 30–49 (1998)
ARTICLE NO. AY985952

Balanced Exponential Growth of Operator Semigroups

Horst R. Thieme*

Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804

Submitted by G. F. Webb

Received April 4, 1997

A C_0-semigroup $S(t)$, $t \geq 0$, on a Banach space X (weakly, strongly, uniformly) approaches *balanced* (*or asynchronous*) *exponential growth* if there exists some $s \in \mathbb{R}$ such that

$$P = \lim_{t \to \infty} e^{-st} S(t)$$

exists (in the weak, strong, uniform operator topology) and P is not the 0 operator. In this paper, the strong and uniform approach to balanced exponential growth is characterized and applicable sufficient conditions are derived. The results will be applied to models for age-size-structured population dynamics in a forthcoming publication. © 1998 Academic Press

Key Words: semigroups; dual semigroups; essentially norm-continuous semigroups; Cauchy problems; positive perturbations; balanced (asynchronous) exponential growth.

1. INTRODUCTION

A C_0-semigroup $S(t)$, $t \geq 0$, on a Banach space X (weakly, strongly, uniformly) approaches *balanced* (*or asynchronous*) *exponential growth* if there exists some $s \in \mathbb{R}$ such that

$$P = \lim_{t \to \infty} e^{-st} S(t)$$

exists (in the weak, strong, uniform operator topology) and P is not the 0 operator. Somewhat shorter we call such a semigroup (weakly, strongly, uniformly) *exponentially balancing*. If S strongly approaches balanced exponential growth and $x \in X$, $Px \neq 0$, then

$$\frac{1}{\|S(t)x\|} S(t)x \to \frac{1}{\|Px\|} Px, \qquad t \to \infty.$$

* Research partially supported by NSF Grant DMS-9403884. E-mail: h.thieme@asu.edu.

0022-247X/98 $25.00

In applications to population dynamics, S describes the unconstrained development of a population, the structure of which (induced by age, spatial distribution, body size, etc.) is contained in the Banach space X. Approach to balanced exponential growth of S means that the population size eventually grows in an exponential fashion and that the normalized structural distribution of the population converges to a final distribution. For this reason we call P the *final distribution operator*. The parameter s is called the intrinsic rate of natural increase (e.g., Metz and Diekmann [32]), or the intrinsic growth constant (Webb [44]), or, in honor of Thomas Malthus [30], the *Malthusian* parameter of S (Jagers [23, Chap. 1] and Iannelli [22]). For the biological importance of balanced exponential growth, see for example, Bell [4], Painter and Marr [34], and Jagers [24].

If S is uniformly exponentially balancing with a final distribution operator P of finite rank, then S is said to have *asynchronous exponential growth* according to Webb [44] who presents necessary and sufficient conditions as well as examples and references for this strongest approach to balanced exponential growth.

Applicable general conditions for asynchronous exponential growth involve quasi-compactness and, often, positivity or even irreducibility of the semigroup (which acts on an ordered Banach space or even a Banach lattice); see Greiner and Nagel [14] and Heijmans and de Pagter [20, Theorem 9.11]. The theory of strongly or weakly exponentially balancing semigroups seems to be less developed. The analogous topic has been widely investigated in the framework of general branching processes, however (see Jagers [25] for a recent account).

If a semigroup S is strongly or weakly exponentially balancing with Malthusian parameter s and final distribution operator P, then P is a projection and

$$e^{-st}S(t)(I-P) \to 0, \qquad t \to \infty,$$

strongly or weakly. In other words, the C_0-semigroup $S_2(t) = e^{-st}S(t)$ restricted to the invariant subspace $X_2 = (I-P)X$ is strongly or weakly asymptotically stable. Sufficient conditions for strong or weak asymptotic stability of C_0-semigroups (that come close to being necessary) have been given by Arendt and Batty [1], Lyubich and Vũ [29], Batty and Vũ [3], and (in a more general framework) Arendt and Prüss [2]; we will use these results frequently (see also van Neerven [42, Chap. 5]). An example of a strong but not uniform approach to balanced exponential growth can be found in Diekmann *et al.* [8], where the reduction to a scalar integral equation and the application of Feller's [9] renewal theorem is possible.

If a C_0-semigroup is weakly exponentially balancing with Malthusian parameter s, then necessarily $s = s(A)$, where A is the infinitesimal generator of S and $s(A)$ the spectral bound of A. Actually, s is necessarily an eigenvalue of both A and A^* and dominates the spectrum of A and *strictly dominates* the point and residual spectrum of A: $s \geq \Re\lambda$ for all spectral values λ of A and $s > \Re\lambda$ for all other eigenvalues λ of A or A^*. If S uniformly approaches balanced exponential growth, then s necessarily is a *strongly dominating* spectral value, i.e., there exists some $\epsilon > 0$ such that $s - \epsilon \geq \Re\lambda$ for all other spectral values λ of A. So a typical situation where one would expect a strong, but not uniform approach to balanced exponential growth is given when the spectral bound $s(A)$ is an eigenvalue of A and dominates the rest of the spectrum strictly, but not strongly. Notice that our terminology is different from the one in Greiner [10], but follows the one in Gyllenberg [16], who presents one of the first examples of this situation. For another example and more remarks, see Diekmann *et al.* [8].

Another necessary condition for a semigroup S to weakly approach balanced exponential growth is *minimal exponential boundedness*, i.e., the boundedness of the semigroup $e^{-s(A)t}S(t)$. Actually, finding sufficient conditions for minimal exponential boundedness becomes a major issue because all the results for asymptotic stability of C_0-semigroups we want to use make the (obviously necessary) assumption that the semigroup is bounded. Most of the usual conditions that guarantee that $e^{-s(A)t}S(t)$ is bounded (like S being positive and quasi-compact and $s(A)$ being a first-order pole of the resolvent of A) typically imply that $s(A)$ is a strongly dominating spectral value of A and so lead to a uniform approach to balanced exponential growth and do not cover the cases that the approach is weak or strong, but not uniform. Another well-known condition—S is a positive C_0-semigroup on a Banach lattice and $s(A)$ is an eigenvalue associated with an eigenvector of A that is an interior point of the positive cone—is quite restrictive.

This paper is organized as follows.

In Section 2 we explain the concept of balanced exponential growth in more detail. We give necessary conditions for weak approach and sufficient conditions for strong approach to balanced exponential growth which almost match. Weak approach implies strong approach whenever the peripheral continuous spectrum of the generator A is countable. The latter assumption is necessary in some, but not all situations. We derive a characterization of uniform approach to balanced exponential growth similar to Webb's [44] characterization of asynchronous exponential growth where the final distribution operator has finite rank in addition. Leaving out this additional feature requires the concept of an essentially norm-continuous semigroup rather than that of an essentially compact semigroup.

In Section 3 we present more results on approach to balanced exponential growth with the emphasis on applicability rather than generality.

Positive perturbations of positive semigroups that operate on closed subspaces of abstract L spaces have been considered in Thieme [38] and results concerning their asynchronous exponential growth are presented in Thieme [39]. In forthcoming work (Thieme [40]) conditions are derived for strong and uniform approach to balanced exponential growth in that framework, and applications to age-size-structured population dynamics are presented.

2. BALANCED EXPONENTIAL GROWTH: CONCEPTS AND CHARACTERIZATIONS

In the tradition of Hille and Phillips [21, Definition 8.3.5], a (one-parameter) semigroup on a vector space X is a family of linear transformations $S(t)$, $t > 0$, satisfying

$$S(t+r) = S(t)S(r), \qquad \forall t, r > 0. \tag{2.1}$$

All semigroups S we are going to consider here operate on a Banach space X and will satisfy the extra condition

$$\limsup_{t \searrow 0} \|S(t)x\| < \infty, \qquad \forall x \in X. \tag{2.2}$$

We call these semigroups B_0-*semigroups*. It follows from the uniform boundedness principle and from (2.1) that any B_0-semigroup is exponentially bounded, i.e., there exist $M \geq 1$, $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \qquad \forall t > 0.$$

A semigroup S is called a C_0-semigroup if

$$\|S(t)x - x\| \to 0, \qquad t \searrow 0, x \in X. \tag{2.3}$$

It is then convenient to extend $S(t)$ to $[0, \infty)$ by

$$S(0)x = x, \qquad x \in X. \tag{2.4}$$

Formula (2.1) then holds for all $t, r \geq 0$ and $S(t)$ is strongly continuous in $t \geq 0$. For C_0-semigroups the infinitesimal generator A is defined by

$$Ax = \lim_{t \searrow 0} \frac{1}{t}(S(t)x - x), \tag{2.5}$$

with $D(A)$ consisting of all elements $x \in X$ for which this limit exists.

Any C_0-semigroup is a B_0-semigroup and so exponentially bounded. If S is a B_0-semigroup on a Banach space X, one introduces the space

$$X_\circ = \{x \in X; \|S(t)x - x\| \to 0, t \searrow 0\}. \tag{2.6}$$

$X_\circ$ is a closed subspace of X that is invariant under $S(t)$ for all $t \geq 0$. The restriction of S to $X_\circ$, $S_\circ$, is a C_0-semigroup on $X_\circ$.

If S is a C_0-semigroup on X, we can consider the dual semigroup $S^*(t) = (S(t))^*$ on the dual Banach space X^*. The uniform boundedness principle implies that S^* is a B_0-semigroup. The space $(X^*)_\circ$ defined by (2.6) for S^* rather than S coincides with $\overline{D(A^*)}$ and the symbol $X^\odot$ is used:

$$X^\odot = \overline{D(A^*)} = (X^*)_\circ.$$

The restriction of the dual semigroup S^* to $X^\odot$ is denoted by $S^\odot$ and the infinitesimal generator of the C_0-semigroup $S^\odot$, $A^\odot$, is the part of A^* in $X^\odot$. One can continue this procedure and consider $X^{\odot *}$ and its closed subspace $X^{\odot\odot} = \overline{D(A^{\odot *})}$ and the C_0-semigroup $S^{\odot\odot}$ on $X^{\odot\odot}$ generated by the part of $A^{\odot *}$ in $X^{\odot\odot}$, $A^{\odot\odot}$. See van Neerven [41] for details and references.

DEFINITION 2.1. Let S be a C_0-semigroup on a Banach space X with infinitesimal generator A. Then S is called (weakly, strongly, uniformly) *balancing* if

$$P = \lim_{t \to \infty} S(t)$$

exists (in the weak, strong, uniform operator topology) and P is not the 0 operator. P is called the *final distribution operator*.

S is called (weakly, strongly, uniformly) *exponentially balancing*—or is said to (weakly, strongly, uniformly) approach *balanced exponential growth*—if there exists some $s \in \mathbb{R}$ such that $e^{-st}S(t)$ is (weakly, strongly, uniformly) balancing. s is called the *Malthusian parameter* of S.

We will soon see that the study of balanced exponential growth of S can be reduced to discussing the conditions for S to be balancing.

We relax the concept of weakly balanced exponential growth as follows.

DEFINITION 2.2. A C_0-semigroup S is called $\odot$ balancing (pronounce "sun balancing") if for any $x \in X$ there exists an element $Px \in X$ such that

$$\langle Px, x^\odot \rangle = \lim_{t \to \infty} \langle S(t)x, x^\odot \rangle, \qquad \forall x^\odot \in X^\odot,$$

and if Px is not 0 for at least one $x \in X$.

We say that S is *exponentially* $\odot$ *balancing* if $e^{-st}S(t)$ is $\odot$ balancing.

Since $X^{\odot}$ separates points in X, Px is uniquely determined and a linear operator P is defined this way which is again called the final distribution operator.

Apparently $\odot$-balanced exponential growth is the weakest concept. We collect some conditions that must necessarily be satisfied for $\odot$-balanced exponential growth to hold. We recall that the spectrum of a densely defined closed operator A is the union of point, residual, and continuous spectra,

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A). \tag{2.7}$$

There seems to be overall agreement in the literature that the point spectrum consists of the eigenvalues of A, i.e., of those λ for which the null space or kernel of $(\lambda - A)$, $N(\lambda - A)$, is not just the 0 space. The other parts of the spectrum are defined differently in the literature. We follow the approach of Hille and Phillips [21] and Pazy [35], e.g., which makes the union in (2.7) disjoint, and let the residual spectrum $\sigma_r(A)$ consist of those complex numbers λ for which $N(\lambda - A) = \{0\}$, but $(\lambda - A)D(A)$ is not dense in X. This way we obtain

$$\sigma_r(A) \subseteq \sigma_p(A^*) \subseteq \sigma_r(A) \cup \sigma_p(A). \tag{2.8}$$

We let the continuous spectrum $\sigma_c(A)$ consist of those λ for which $N(\lambda - A) = \{0\}$ and

$$(\lambda - A)D(A) \neq \overline{(\lambda - A)D(A)} = X.$$

Both the point and the continuous spectra are contained in the approximate point spectrum $\sigma_a(A)$, which consists of those complex numbers λ for which there exists a sequence $(x_n) \in D(A)$ such that

$$\|x_n\| = 1, \qquad (\lambda - A)x_n \to 0, \qquad n \to \infty. \tag{2.9}$$

As usual, $s(A)$ denotes the spectral bound of A,

$$s(A) = \sup \Re\sigma(A) = \sup\{\Re\lambda;\ \lambda \in \sigma(A)\}. \tag{2.10}$$

If $s(A) \in \mathbb{R}$, we call

$$\sigma^{\#}(A) = \sigma(A) \cap (s(A) + \iota\mathbb{R}) \tag{2.11}$$

the *boundary*, or *peripheral*, or *principal spectrum* of A. In other words, the principal spectrum consists of the spectral values with maximum real part.

Analogously, we define the *principal point*, *residual*, *and continuous spectra* of A, which are denoted by $\sigma_p^{\#}(A)$, $\sigma_r^{\#}(A)$, and $\sigma_c^{\#}(A)$, respectively. We have

$$\sigma^{\#}(A) \subseteq \partial\sigma(A) \subseteq \sigma_a(A) \tag{2.12}$$

(Greiner and Nagel [13, Prop. 2.2]). If A is the infinitesimal generator of a C_0-semigroup, then Lemma 2.3 in Arendt and Batty [1], with our partition of the spectrum which is different from theirs, reads

$$\sigma_p^{\#}(A) \cup \sigma_r^{\#}(A) = \sigma_p^{\#}(A^*). \tag{2.13}$$

PROPOSITION 2.3. *Let S be a C_0-semigroup on a Banach space X that is exponentially $\odot$ balancing with Malthusian parameter s and final distribution operator P. Then*

$$\|S(t)\| \le Me^{st}, \qquad t \ge 0,$$

with some constant $M > 0$ and $s = s(A)$. Further

$$\sigma_p^{\#}(A) \cup \sigma_r^{\#}(A) = \sigma_p^{\#}(A^*) = \{s\},$$

and s is an eigenvalue of A. Moreover, P is a bounded projection from X onto $N(A - s)$. Finally,

$$(\lambda - s)(\lambda - A)^{-1} \to P, \qquad \lambda \searrow s,$$

in the strong operator topology.

Proof. Without restricting the generality, we can assume that $s = 0$. The boundedness of S follows from the uniform boundedness principle and the fact that $X^{\odot}$ is a norming subspace of X^* for X, i.e., the norm $\|\cdot\|'$ defined by

$$\|x\|' = \sup\{|\langle x, x^{\odot}\rangle|;\ x^{\odot} \in X^{\odot},\ \|x^{\odot}\| \le 1\} \tag{2.14}$$

is equivalent to the original norm on X. See Clément [5, Sect. 3.3]. This same fact implies that the linear operator P is bounded. Obviously, neither A nor A^* are allowed to have purely imaginary eigenvalues. Now

$$\langle PS(r)x, x^{\odot}\rangle = \lim_{t\to\infty} \langle S(t)S(r)x, x^{\odot}\rangle = \lim_{t\to\infty} \langle S(t+r)x, x^{\odot}\rangle = \langle Px, x^{\odot}\rangle.$$

Since $X^{\odot}$ separates points in X, we have $PS(r) = P$. Further

$$\begin{aligned}\langle S(r)Px, x^{\odot}\rangle &= \langle Px, S^{\odot}(r)x^{\odot}\rangle = \lim_{t\to\infty} \langle S(t)x, S^{\odot}(r)x^{\odot}\rangle \\ &= \lim_{t\to\infty} \langle S(t+r)x, x^{\odot}\rangle = \langle Px, x^{\odot}\rangle.\end{aligned}$$

Since $X^{\odot}$ separates points in X, $S(r)P = P$. This implies $P^2 = P$. Since P is not the 0 operator, 0 is an eigenvalue of A and P maps into $N(A)$. On the other hand, if $x \in N(A)$, $S(r)x = x$ for all $r \geq 0$ and so $Px = x$. Taking Laplace transforms, we find

$$\lambda(\lambda - A)^{-1} P = P = P\lambda(\lambda - A)^{-1}, \qquad \forall \lambda > 0.$$

Further

$$\langle Px, x^{\odot} \rangle = \lim_{\lambda \searrow 0} \langle \lambda(\lambda - A)^{-1} x, x^{\odot} \rangle.$$

Hence, for any $x^* \in X^*$,

$$\begin{aligned} \langle Px, x^* \rangle &= \langle \mu(\mu - A)^{-1} Px, x^* \rangle = \langle Px, \mu(\mu - A)^{-1*} x^* \rangle \\ &= \lim_{\lambda \searrow 0} \langle \lambda(\lambda - A)^{-1} x, \mu(\mu - A)^{-1*} x^* \rangle \\ &= \lim_{\lambda \searrow 0} \langle \lambda\mu(\mu - A)^{-1}(\lambda - A)^{-1} x, x^* \rangle. \end{aligned}$$

By the resolvent identity,

$$\begin{aligned} \langle Px, x^* \rangle &= \lim_{\lambda \searrow 0} \left\langle \frac{\lambda\mu}{\mu - \lambda}\left((\lambda - A)^{-1} - (\mu - A)^{-1}\right) x, x^* \right\rangle \\ &= \lim_{\lambda \searrow 0} \langle \lambda(\lambda - A)^{-1} x, x^* \rangle. \end{aligned}$$

Convergence in the strong rather than in the weak operator topology follows from Hille and Phillips [21, Theorem 18.7.3].

The next result in particular implies that an exponentially $\odot$ balancing C_0-semigroup is strongly exponentially balancing provided that the principal spectrum is countable.

THEOREM 2.4. *Let S be a C_0-semigroup on the Banach space X with infinitesimal generator A. Then the following set of conditions is necessary for S to be exponentially $\odot$ balancing. This set of conditions is sufficient for S to be strongly exponential balancing if, in addition, the principal spectrum $\sigma^{\#}(A)$ of A is countable*:

(1) *$e^{-s(A)t} S(t)$ is a bounded semigroup.*

(2) $\sigma_p^{\#}(A^*) = \{s(A)\}$.

(3) *For any $x \in X$, the sets*

$$\left\{(\lambda - s(A))(\lambda - A)^{-1} x;\ s(A) < \lambda < s(A) + 1\right\}$$

are sequentially weakly compact in X.

Condition (3) can equivalently be expressed as $R(A - s(A)) + N(A - s(A))$ being dense in X or as $N(A - s(A))$ separating $N(A^* - s(A))$ (Hille and Phillips [21, Theorem 18.7.3]). It is, in particular, satisfied if $(\mu - A)^{-1}$ is weakly compact for some $\mu \in \rho(A)$ which, in turn, is equivalent to X being sun-reflexive (de Pagter [7]).

Proof. It follows from Proposition 2.3 that these conditions are necessary for approach to $\odot$-balanced exponential growth. Without restricting the generality we assume that $s(A) = 0$ and show that conditions (1)–(3) are sufficient for S to be strongly balancing provided that $\sigma^{\#}(A)$ is countable. By condition (3), the limit

$$P = \lim_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}$$

exists strongly (Hille and Phillips [21, Theorem 18.7.3]) and defines a projection P onto $N(A)$,

$$S(t)P = P = PS(t), \qquad t \geq 0.$$

We have $X = X_1 \oplus X_2$ with $X_1 = PX$, $X_2 = (I - P)X$, and both X_1 and X_2 are invariant under S. Let S_1, S_2 be the restrictions of S to X_1, X_2 respectively and A_1, A_2 their respective infinitesimal generators. Then $S_1(t) = I$ on X_1 and $\sigma_p(A_2^*) = \sigma_p(A^*) \setminus \{0\}$; hence $\sigma_p^{\#}(A_2^*) = \emptyset$. Hence $S_2(t) \to 0$ strongly on X_2 as $t \to \infty$ (Arendt and Batty [1] or Ljubich and Vũ [29]; see also van Neerven [42, Sect. 5.1]). Thus $S(t)x = S_1(t)Px + S_2(t)(I - P)x \to Px$ as $t \to \infty$, $x \in X$.

At the end of this section we shall see that it is not possible in general to drop the assumption that the principal spectrum is countable. But we shall also see that this assumption is not necessary either. A complete, but perhaps less applicable characterization of the approach to balanced exponential growth follows from the Jacobs–Deleeuw–Glicksberg splitting theorems (Krengel [28, Sect. 2.1, Theorems 4.4 and 4.5]).

THEOREM 2.5. *Let S be a C_0-semigroup on the Banach space X, A its infinitesimal generator, and $s = s(A)$ the spectral bound of A. Then S is strongly exponentially balancing if and only if the following conditions hold*:

(1) *The semigroup $e^{-st}S(t)$ is strongly almost periodic, i.e., all orbits $\{e^{-st}S(t)x;\ t \geq 0\}$, $x \in X$, are precompact.*

(2) $\sigma_p^{\#}(A) = \{s\}$.

Proof. We base our proof on a result by Jamison [26]. Without restricting the generality we assume that $s(A) = 0$. Apparently conditions (1) and (2) are necessary for S to be balancing. In order to see that they are sufficient, let $r > 0$ be fixed, but arbitrary, and consider $K = S(r)$. Then K

has no eigenvalue μ with $|\mu| = 1$ and $\mu \neq 1$. Further, for any $x \in X$, the orbit $\{K^n x;\ n \in \mathbb{N}\}$ is precompact. Then

$$x_r = \lim_{n\to\infty} K^n x = \lim_{n\to\infty} S(nr)x$$

exists in norm (Jamison [26, Theorem 3.2]). But x_r is contained in the compact closure of the orbit $\{S(t)x;\ t \geq 0\}$. Hence there exists a sequence $r_n \searrow 0$ and some $x_0 \in X$ such that $x_{r_n} \to x_0$ as $n \to \infty$. Since S is a C_0-semigroup, it follows that $S(t)x \to x_0$ as $t \to \infty$. By condition (2), $P = \lim_{t\to\infty} S(t)$ is not the 0 operator.

Finally we turn to a characterization of C_0-semigroups that uniformly approach balanced exponential growth. As we will see in the next section, the following class contains both essentially compact semigroups and eventually norm-continuous semigroups.

DEFINITIONS 2.6. A C_0-semigroup S is called *essentially norm-continuous if*

$$S(t) = U_1(t) + U_2(t)$$

with two operator families U_1, U_2 such that U_2 is operator-norm right-continuous and there exists some $\gamma < \omega(S)$ such that

$$e^{-\gamma t}\|U_1(t)\| \text{ is bounded.}$$

We could make Definition 2.6 look more general by only requiring that U_2 is eventually norm right-continuous, i.e., norm right-continuous in $t \geq t_0$ with some $t_0 \geq 0$. But we can always achieve the above situation by redefining U_1, U_2. An essentially norm-continuous semigroup is norm-continuous at infinity (Martinez and Mazon [31]).

THEOREM 2.7. *Let S be a C_0-semigroup with infinitesimal generator A. Then S is uniformly exponentially balancing if and only if the following set of conditions holds*:

(1) *S is essentially norm-continuous.*

(2) $\sigma^{\#}(A) = \{s(A)\}$.

(3) *$s(A)$ is a first-order pole of the resolvent of A.*

If S is a positive C_0-semigroup in a Banach lattice, condition (2) is redundant (see Theorem 3.4).

We will see that $s(A)$ is a strongly dominating spectral value, i.e., the rest of the spectrum is bounded away from the line $s(A) + \iota\mathbb{R}$.

Proof. Without restricting the generality we can assume that $s(A) = 0$. Let us first assume that S is uniformly balancing. By Proposition 2.3, S is bounded and $s(A) = 0 = \omega(S)$. We have

$$S(t) \to P, \qquad t \to \infty,$$

in operator norm. It follows that $S(r)P = P = PS(r)$ for all $r \geq 0$ and that $U_1(t) := S(t)(I - P) \to 0$ as $t \to \infty$ in operator norm. Moreover, $U_1(t)$, $t \geq 0$, is a semigroup (satisfying $U_1(0) = I - P$). Hence there exists some $\epsilon > 0$ such that

$$e^{\epsilon t}\|U_1(t)\| \to 0, \qquad t \to \infty,$$

Further, $U_2(t) = S(t) - U_1(t) = S(t)P = P$ is norm-continuous. This implies that S is essentially norm-continuous. Further

$$\lambda(\lambda - A)^{-1} \to P, \qquad \lambda \to 0,$$

in operator norm. Since P is not the 0 operator, 0 is a first-order pole of the resolvent of A (Hille and Phillips [21, Theorem 18.8.1]). Since P maps X onto $N(A)$ and $S(t)(I - P) \to 0$ as $t \to \infty$, we see that 0 is a strongly dominating eigenvalue, i.e., $\sigma(A) \setminus \{0\}$ is bounded away from the imaginary axis. In particular, 0 is the only purely imaginary spectral value.

Let us now assume that conditions (1)–(3) are satisfied with $s(A) = 0$. Then S is norm-continuous at infinity and $\omega(S) = s(A) = 0$ by Corollary 1.4 in Martinez and Mazon [31]. Hence the number γ in Definition 2.6 can be chosen strictly negative. By Theorem 1.9 in Martinez and Mazon [31], there exists some $\epsilon > 0$ such that the set

$$\{\lambda \in \sigma(A); \Re\lambda > -\epsilon\}$$

is bounded. Conditions (2) and (3) now imply that there exists some $\epsilon > 0$ such that

$$\Re\lambda < -\epsilon, \qquad \lambda \in \sigma(A) \setminus \{0\}.$$

Since 0 is a first-order pole,

$$P = \lim_{\lambda \to 0} \lambda(\lambda - A)^{-1}$$

exists in operator norm and maps X onto $N(A)$. Moreover, $S(t)P = P = PS(t)$ and the closed subspaces $X_1 = PX$, $X_2 = (I - P)X$ are invariant under S. Let S_1, S_2 be the C_0-semigroups on X_1, X_2 obtained by restricting S to these invariant subspaces. The infinitesimal generators A_1, A_2 of S_1, S_2 are the parts of A in X_1, X_2, respectively. Then $\sigma(A_2) = \sigma(A) \setminus \{0\}$ (Kato [27, III, Sect. 6.4, Theorem 6.17]). Since 0 strongly dominates the rest of $\sigma(A)$, $s(A_2) < 0$. Suppose that the type of S_2 satisfies $\omega(S_2) \geq 0$.

As we remarked before, the number γ in Definition 2.6 can be chosen strictly negative. Hence S_2 is essentially norm-continuous and thus also norm-continuous at infinity. Further $\operatorname{spr} S_2(t) \geq 1$ for all $t \geq 0$. By Theorem 1.2 in Martinez and Mazon [31], $s(A_2) \geq 0$, a contradiction. Hence $\|S_2(t)\| \to 0$ as $t \to \infty$. Thus $S(t) = S(t)P + S_2(t)(I - P) = P + S_2(t)(I - P) \to P$ in operator norm as $t \to \infty$.

We conclude this section by illustrating the various approaches to balanced exponential growth in the context of multiplication semigroups. We start with sequence spaces.

Let (λ_m) be a sequence of complex numbers. Consider the vector space $E = \mathbb{R}^{\mathbb{N}}$ and define a semigroup on E by

$$(S(t)x)(m) = e^{\lambda_m t} x(m), \qquad x \in \mathbb{R}^{\mathbb{N}}, m \in \mathbb{N}. \tag{2.15}$$

On $\mathbb{R}^{\mathbb{N}}$ we have the product topology (topology of coordinate-wise convergence) and S is a continuous semigroup with respect to it. We can define what it means that S is balancing with respect to the product topology, namely that

$$\lim_{t \to \infty} (S(t)x)(m)$$

exists for every $x \in \mathbb{R}^{\mathbb{N}}$, $m \in \mathbb{N}$. Apparently this limit, if it exists, defines a projection P on $\mathbb{R}^{\mathbb{N}}$ such that $S(t)P = P = PS(t)$. As before, we require P not to be the 0 projection. We then say S is *coordinate-wise balancing* and we call P the final distribution operator. It is easy to prove the following result:

EXAMPLE 2.8. The multiplication semigroup (2.15) on $\mathbb{R}^{\mathbb{N}}$ is coordinate-wise balancing if and only if the following conditions hold:

- $\Re \lambda_m \leq 0$ for all $m \in \mathbb{N}$.
- *No* λ_m is purely imaginary.
- $\lambda_m = 0$ for some $n \in \mathbb{N}$.

If S is coordinate-wise balancing, the final distribution operator is given by

$$(Px)(m) = \begin{cases} 0, & \Re \lambda_m < 0, \\ x(m), & \lambda_m = 0. \end{cases}$$

We now consider the multiplication semigroup S on the various Banach sequence spaces. We keep the topology of coordinate-wise convergence by restricting the product topology to the respective Banach sequence space

X. Apparently S is coordinate-wise balancing if S is $\odot$ balancing. Moreover, Example 2.8 remains valid. The associated infinitesimal generator is given by

$$(Ax)(m) = \lambda_m x(m),$$

with $D(A)$ consisting of those $x \in X$ for which Ax is in X again. Apparently the λ_m are the eigenvalues of A.

EXAMPLE 2.9. Let S be a multiplication semigroup on $X = (c_0)$ or $X = l_p$, $1 \le p < \infty$.

(a) S is strongly balancing if and only if it is $\odot$ balancing if and only if it is coordinate-wise balancing.

(b) S is uniformly balancing if and only if the following two conditions hold:

- $\sup\{\Re\lambda_m;\ m \in \mathbb{N},\ \lambda_m \neq 0\} < 0$.
- $\lambda_m = 0$ for some $m \in \mathbb{N}$.

Remark 2.10. (a) This example shows that there are C_0-semigroups that are strongly balancing, but not uniformly balancing.

(b) In view of Theorem 2.4, we learn that the countability of the principal spectrum is not necessary for a semigroup to be strongly balancing. Actually, we can construct a strongly balancing semigroup where the principal spectrum of the generator contains the whole imaginary axis. Let ϕ: $\mathbb{N} \to \mathbb{Q}$ be a mapping from the natural number onto the rational numbers. Set

$$\kappa(j,k) = -\frac{1}{j} + \iota\phi(k).$$

Let ψ be a mapping from the $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$ and set

$$\lambda_m = \begin{cases} \kappa(\psi(m-1)), & m \ge 2, \\ 0, & m = 1. \end{cases}$$

Then λ_m satisfies the assumptions of Example 2.9(a) and

$$\sigma(A) = \overline{\{\lambda_m;\ m \in \mathbb{N}\}} \supseteq \iota\mathbb{R}.$$

Similarly, we can construct a semigroup that is asymptotically stable, but where the spectrum of its generator contains the whole imaginary axis.

It is also instructional to look at multiplication semigroups on $(c) = c(\mathbb{N})$, the space of convergent sequences with the supremum norm. We recall that the dual space of (c) can be identified with $l_1(\mathbb{N}_0)$ where

$$\langle x, x^* \rangle = x_0^* \lim_{m\to\infty} x_m + \sum_{m=1}^{\infty} x_m x_m^*, \qquad x \in (c),\ x^* \in l_1(\mathbb{N}_0).$$

For a multiplication semigroup to map (c) into (c), it is necessary and sufficient that

$$\lambda_0 = \lim_{m\to\infty} \lambda_m$$

exists. Hence

$$\sigma(A) = \{\lambda_m; m \in \mathbb{N}_0\}.$$

In particular, the spectrum of A is countable and so S is strongly balancing if and only if S is $\odot$ balancing. The dual semigroup on $l_1(\mathbb{N}_0)$ is given by

$$(S^*(t)x^*)(m) = e^{\lambda_m t}x^*(m), \qquad m \in \mathbb{N}_0.$$

Obviously, the dual semigroup S^* is a C_0-semigroup if $\Re\lambda_m \le 0$ for all $m \in \mathbb{N}$.

EXAMPLE 2.11. Let S be a multiplication semigroup on (c). Then the following statements are equivalent:

(i) S is $\odot$ balancing.

(ii) S is strongly balancing.

(iii) S is uniformly balancing.

(iv) S is coordinate-wise balancing and $\lim_{m\to\infty} \Re\lambda_m = \Re\lambda_0 < 0$ or $\lambda_m = 0$ for almost all m.

Remark 2.12. Let S on (c) be coordinate-wise balancing and $\lambda_0 = \lim_{m\to\infty}\lambda_m = 0$, but $\lambda_m \neq 0$ for infinitely many m. Then the semigroup S^* on $l_1(\mathbb{N}_0)$ is strongly balancing, but S on (c) is not $\odot$ balancing.

We have learned in Remark 2.10(b) that the countability of the principal spectrum of A is not necessary for a semigroup to be strongly balancing. The last example, which is a modification of Example 2.5 in Arendt and Batty [1], shows that it is not possible to drop this assumption in general.

EXAMPLE 2.13. Let $X = L^2(0,1) \times \mathbb{R}$ and

$$S(t)(f,\alpha) = (e^{\iota t\cdot}f, \alpha), \qquad f \in L^2(0,1),\ \alpha \in \mathbb{R}.$$

Then S is a bounded C_0-semigroup with $\sigma_p(A) = \sigma_r(A) = \{0\}$, $\sigma_c(A) = \{\iota\nu; 0 < \nu < 1\}$. Moreover,

$$\lambda(\lambda - A)^{-1}(f,\alpha) \to (0,\alpha), \qquad \lambda \to 0, (f,\alpha) \in X.$$

So conditions (1)–(3) in Theorem 2.4 are satisfied, but S is not strongly (or even $\odot$) balancing.

3. MORE RESULTS ON BALANCED EXPONENTIAL GROWTH

We start with uniform approach to balanced exponential growth. We recall the definition of asynchronous exponential growth (Webb [44]).

DEFINITION 3.1. A C_0-semigroup exhibits *asynchronous exponential growth* if it is uniformly exponentially balancing and the finite distribution operator P has finite-dimensional range.

We recall that a uniformly exponential balancing semigroup necessarily is essentially norm-continuous (Theorem 2.7). Essentially norm-continuous semigroups provide a unifying framework for two classes of semigroups that have been studied before: eventually norm-continuous semigroups and essentially compact C_0-semigroups.

We recall that a C_0-semigroup is eventually norm-continuous if $S(t)$, $t \geq r$, is norm-continuous for some $r \geq 0$. S is essentially compact if the essential type of S is strictly smaller than the type of S. Equivalently,

$$\operatorname{dist}\left(e^{-\omega(S)t} S(t), \mathscr{K}\right) \to 0, \qquad t \to \infty,$$

where $\mathscr{K}$ denotes the ideal of compact operators on X (Greiner [11, Prop. 2.8]).

PROPOSITION 3.2. *Eventually norm-continuous semigroups and essentially compact semigroups are essentially norm-continuous.*

Proof. Let $S(t)$, $t \geq r$, be norm-continuous for some $r > 0$. Setting

$$U_2(t) = \begin{cases} S(r), & 0 \leq t \leq r, \\ S(t), & t \geq r, \end{cases}$$

and $U_1(t) = S(t) - U_2(t)$, we find that $U_1(t) = 0$ for $t \geq r$ and U_2 is norm-continuous.

Let S be essentially compact. The proof of Proposition 2.8 in Greiner [11] shows that we can actually find some $\epsilon > 0$ such that

$$\operatorname{dist}\left(e^{(\epsilon - \omega(S))t} S(t), \mathscr{K}\right) \to 0, \qquad t \to \infty.$$

Without restricting the generality we assume that $\omega(S) = 0$. Then $s(A) = 0$ (Heijmans [17, Prop. 8.6]). Further we find compact linear operators K_n, $n \in \mathbb{N}$, and a constant $M \geq 1$, such that

$$\|S(n) - K_n\| \leq Me^{-\epsilon n}, \qquad n \in \mathbb{N}.$$

We set

$$U_2(t) = \begin{cases} I, & 0 \leq t < 1, \\ S(t-n)K_n, & n \leq t < n+1, n \in \mathbb{N}, n \geq 1. \end{cases}$$

Since the operators K_n are compact, $U_2(t)$ is right-continuous in operator norm. Moreover, we have that, for $n \leq t < n + 1$,

$$U_1(t) = S(t) - U_2(t) = S(t-n)(S(n) - K_n).$$

Hence

$$\|U_1(t)\| \leq \|S(t-n)\| \, \|S(n) - K_n\| \leq \sup_{0 \leq r \leq 1} \|S(r)\| M e^{-\epsilon n}$$

$$\leq \sup_{0 \leq r \leq 1} \|S(r)\| M e^{\epsilon(1-t)}.$$

Thus S is essentially norm-continuous.

The following result which is an immediate consequence of Theorem 2.7 can now be compared with the characterization of asynchronous exponential growth by Webb [44, Prop. 2.3].

THEOREM 3.3. *Let S be a C_0-semigroup with infinitesimal generator A. Then S exhibits asynchronous exponential growth if and only if the following set of conditions holds*:

(1) *S is essentially norm-continuous.*

(2) $\sigma^{\#}(A) = \{s(A)\}$.

(3) *$s(A)$ is a first-order pole of the resolvent of A with finite multiplicity.*

Remark. Essential norm-continuity in condition (1) can be replaced by norm-continuity at infinity.

Proof. If S is norm-continuous at infinity, we can argue with Theorem 1.9 of Martinez and Mazon [31] as in the proof of Theorem 2.7 that there exists some $\epsilon > 0$ such that $\Re \lambda < s(A) - \epsilon$ for all spectral values λ of A except $s(A)$. Since $s(A)$ is a pole of the resolvent with finite algebraic multiplicity, this implies $s_{\text{ess}}(A) < 0$. Hence S is essentially compact by Theorem 1.5 in Martinez and Mazon [31]. Thus $\omega_{\text{ess}}(S) < \omega(S) = s(A)$ with the last equality following from Corollary 1.4 in Martinez and Mason [31]. Asynchronous exponential growth of S now follows from Proposition 2.3 in Webb [44].

We turn to positive semigroups on Banach lattices.

THEOREM 3.4. *Let X be a Banach lattice and S a positive C_0-semigroup with generator A. Then S is uniformly exponentially balancing if and only if S is essentially norm-continuous and $s(A)$ is a first-order pole of the resolvent of A.*

Proof. By Theorem 2.7, we need to show that $s(A)$ is the only point in the principal spectrum of A, if $s(A)$ is a first-order pole. If $s(A)$ is a first-order pole, the resolvent of A is slowly growing and the principal spectrum is cyclic (Greiner [12, Prop. 2.9 and Theorem 2.10]). This implies that the principal spectrum either consists of $s(A)$ or contains all points $s(A) + \iota k \nu$, $k \in \mathbb{Z}$, for some $\nu > 0$. On the other hand, the principal spectrum is bounded because S is norm-continuous at infinity (Martinez and Mazon [31, Theorem 1.9]). Hence $s(A)$ is the only principal spectral value.

We finally turn to balanced exponential growth for positive semigroups on Banach lattices.

THEOREM 3.5. *Let S be a positive C_0-semigroup on a Banach lattice X with infinitesimal generator A such that the spectral bound of A, $s(A)$, is a pole of the resolvent of S with finite geometric multiplicity. Then S, $S^{\odot}$, and $S^{\odot\odot}$ are strongly exponentially balancing with Malthusian parameter $s(A)$ and final distribution operator of finite rank if and only if the following hold*:

(1) $e^{-s(A)t}S(t)$ *is a bounded semigroup.*

(2) *There exists no $\nu > 0$ such that, for every $k \in \mathbb{Z}$, $s(A) + \iota k \nu$ is a first-order pole of the resolvent of A with finite multiplicity.*

Remark. If $s(A)$ is a pole of the resolvent of A whose geometric multiplicity is not necessarily finite, we can replace condition (2) by

(2′) There exists no $\nu > 0$ such that

$$s(A) + \iota k \nu \in \sigma(A), \qquad \forall k \in \mathbb{Z}.$$

We do not obtain the final distribution operator to be of finite rank, though, and condition (2′) does not seem to be a necessary condition.

Proof. The necessity of these conditions follows from Proposition 2.3. In order to show that they are sufficient, we assume that $s(A) = 0$ and show that S, $S^{\odot}$, and $S^{\odot\odot}$ are strongly balancing. Since S is a bounded semigroup, 0 is a first-order pole. Since geometric and algebraic multiplicity of a first-order pole coincide, 0 has finite algebraic multiplicity. Then $\sigma^{\#}(A)$ is cyclic, i.e., if $\nu > 0$ and $\iota\nu \in \sigma(A)$, then $\iota k \nu \in \sigma(A)$ for all $k \in \mathbb{Z}$ (Greiner [12, Prop. 2.9 and Theorem 2.10] and Heijmans [17, Theorem 8.14]). Moreover, $s(A) + \iota k \nu$ is a pole of finite algebraic multiplicity (Greiner [12, Theorem 3.14]). The order of $s(A) + \iota k \nu$ does not exceed the order of 0 (Heijmans [17, Theorem 8.7]), and so $s(A) + \iota k \nu$ is a first-order pole. From condition 2 we conclude that $\sigma^{\#}(A) = \{0\}$. Since 0 is a first-order pole, the limit

$$P = \lim_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}$$

exists in the uniform operator topology and P is a projection onto the finite-dimensional eigenspace $N(A)$ along $R(A)$. We have $X = N(A) \oplus R(A)$ with both subspaces being invariant under the resolvent of A and so under S. If S_2 is the restriction of S to $R(A)$ and A_2 its infinitesimal generator, then $\sigma(A_2) = \sigma(A) \setminus \{0\}$ (Kato [27, III, Sect. 6.4, Theorem 6.17]). Hence $\sigma(A_2) \cap \iota\mathbb{R} = \emptyset$. This implies that $S_2(t) \to 0$ strongly as $t \to \infty$ (Arendt and Batty [1] or Ljubich and Vũ [29]). As $S(t)x = x$ for $x \in N(A)$, we have $S(t)x \to x$ as $t \to \infty$. The same arguments work for $S^{\odot}$ once we remember that $\sigma(A^{\odot}) = \sigma(A) \ni 0$ and

$$\sigma_p(A^{\odot}) = \sigma_p(A^*) \subseteq \sigma_r(A) \cup \sigma_p(A).$$

Hence

$$\sigma_p(A^{\odot}) \cap \iota\mathbb{R} = \sigma_p(A) \cap \iota\mathbb{R} = \{0\}.$$

Notice that $\sigma_p(A) \cap \iota\mathbb{R} \subseteq \sigma_p(A^*) \cap \iota\mathbb{R}$ (Arendt and Batty [1, Lemma 2.3]). Further $(\lambda - A^{\odot})^{-1}$ is the restriction of $(\lambda - A)^{-1*}$ to $\overline{D(A^*)}$. The same arguments work for $S^{\odot\odot}$.

ACKNOWLEDGMENTS

I thank Odo Diekmann and Mats Gyllenberg for a useful discussion and Jan van Neerven for bringing the paper by Martinez and Mazon [31] to my attention.

REFERENCES

1. W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.* **306** (1988), 837–852.
2. W. Arendt and J. Prüss, Vector-valued Tauberian theorems and asymptotic behavior of linear Volterra equations, *SIAM J. Math. Anal.* **23** (1992), 412–448.
3. C. J. K. Batty and Vũ Quôc Phóng, Stability of individual elements under one-parameter semigroups, *Trans. Amer. Math. Soc.* **322** (1990), 805–818.
4. G. I. Bell, Cell growth and division III. Conditions for balanced exponential growth in a mathematical model, *Biophys. J.* **8** (1968), 431–444.
5. Ph. Clément, Generation of linear semigroups, *in* "One-Parameter Semigroups" (Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, Eds.), pp. 47–92, North-Holland, Amsterdam, 1987.
6. Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, "One-Parameter Semigroups," North-Holland, Amsterdam, 1987.
7. B. de Pagter, A characterization of sun-reflexivity, *Math. Ann.* **283** (1989), 511–518.
8. O. Diekmann, M. Gyllenberg, H. R. Thieme, and S. M. Verduyn Lunel, A cell-cycle model revisited, *in* "Proceedings of Conférence Internationale sur les Equations Differentielles, Marrakech, 1992," to appear.
9. W. Feller, "An Introduction to Probability Theory and Its Applications," Vol. II, Wiley, New York, 1966.

10. G. Greiner, Spectral theory of positive semigroups on $C_0(X)$, *in* "One-Parameter Semigroups of Positive Operators," (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 163–203, Springer-Verlag, New York, 1986.
11. G. Greiner, Compact and quasi-compact semigroups, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 209–218, Springer-Verlag, New York, 1986.
12. G. Greiner, Spectral theory of positive semigroups on Banach lattices, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 292–332, Springer-Verlag, New York, 1986.
13. G. Greiner and R. Nagel, Spectral theory of semigroups on Banach Spaces, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 60–97, Springer-Verlag, New York, 1986.
14. G. Greiner and R. Nagel, Convergence of positive semigroups, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 342–355, Springer-Verlag, New York, 1986.
15. G. Greiner and F. Neubrander, Stability of positive semigroups on Banach lattices, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 334–342, Springer-Verlag, New York, 1986.
16. M. Gyllenberg, The size and scar distribution of the yeast *Saccharomyces cerevisiae*, *J. Math. Biol.* **24** (1986), 81–101.
17. H. J. A. M. Heijmans, Perron–Frobenius theory for positive semigroups, *in* "One-Parameter Semigroups" (Ph. Clément, H. J. A. M. Heijmans, S. Angenent,C. J. van Duijn, and B. de Pagter, Eds.), pp. 193–211, North-Holland, Amsterdam, 1987.
18. H. J. A. M. Heijmans, Two examples from structured population dynamics, *in* "One-Parameter Semigroups" (Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, Eds.), pp. 235–244, North-Holland, Amsterdam, 1987.
19. H. J. A. M. Heijmans, Some results from spectral theory, *in* "One-Parameter Semigroups" (Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, Eds.), pp. 281–291, North-Holland, Amsterdam, 1987.
20. H. J. A. M. Heijmans and B. de Pagter, Asymptotic behavior, *in* "One-Parameter Semigroups" (Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, Eds.), pp. 213–233, North-Holland, Amsterdam, 1987.
21. E. Hille and R. S. Phillips, "Functional Analysis and Semi-Groups," Amer. Math. Soc., Providence, 1957.
22. M. Iannelli, "Mathematical Theory of Age-Structured Population Dynamics," Applied Mathematics Monographs (C.N.R.), Vol. 7, Giardini Editori e Stampatori, Pisa, 1995.
23. P. Jagers, "Branching Processes with Biological Applications," Wiley, New York, 1975.
24. P. Jagers, Balanced exponential growth: What does it mean and when is it there? *in* "Biomathematics and Cell Kinetics, Development in Cell Biology" (A. Valleron and P. MacDonald, Eds.), Vol. 2, pp. 21–29, Elsevier, New York, 1978.
25. P. Jagers, The deterministic evolution of general branching populations, *in* "Mathematical Population Dynamics: Analysis of Heterogeneity" (O. Arino, D. Axelrod, M. Kimmel, and M. Langlais, Eds.), to appear.
26. B. Jamison, Asymptotic behavior of successive iterates of continuous functions under a Markov operator, *J. Math. Anal. Appl.* **9** (1964), 203–214.
27. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York, 1976.
28. U. Krengel, "Ergodic Theorems," De Gruyter, Berlin, 1985.
29. Y. I. Ljubich and Vũ Quôc Phóng, Asymptotic stability of linear differential equations in Banach spaces, *Studia Math.* **88** (1988), 37–42.
30. T. R. Malthus, "An Essay on the Principle of Populations" (and "A Summary View of the Principle of Population"), Penguin, Harmondsworth, Middlesex, London, 1798, 1970.

31. J. Martinez and J. M. Mazon, C_0-semigroups norm continuous at infinity, *Semigroup Forum* **52** (1996), 213–224.
32. J. A. J. Metz and O. Diekmann, "The Dynamics of Physiologically Structured Populations," Lecture Notes in Biomathematics, Vol. 68, Springer-Verlag, New York, 1986.
33. F. Neubrander, Asymptotics of semigroups on Banach spaces, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Mathematics, Vol. 1184, pp. 98–116, Springer-Verlag, New York, 1986.
34. P. R. Painter and A. G. Marr, Mathematics of microbial populations, *Ann. Rev. Microbiol.* **22** (1968), 519–548.
35. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.
36. H. H. Schaefer, "Topological Vector Spaces," Springer-Verlag, New York, 1966.
37. H. H. Schaefer, "Banach Lattices," Springer-Verlag, New York, 1974.
38. H. R. Thieme, Positive perturbations of dual and integrated semigroups, *Adv. Math. Sci. Appl.* **6** (1996), 445–507.
39. H. R. Thieme, Positive perturbation of operator semigroups: Growth bounds, essential compactness, and asynchronous exponential growth, *Disc. Cont. Dyn. Sys.* (to appear).
40. H. R. Thieme, Balanced exponential growth of perturbed operator semigroups, preprint.
41. J. M. A. M. van Neerven, "The Adjoint of a Semigroup of Linear Operators," Lecture Notes in Mathematics, Vol. 1529, Springer-Verlag, New York, 1992.
42. J. M. A. M. van Neerven, "The Asymptotic Behaviour of Semigroups of Linear Operators," Birkhäuser, Basel, 1996.
43. G. F. Webb, "Theory of Nonlinear Age-Dependent Population Dynamics," Dekker, New York, 1985.
44. G. F. Webb, An operator-theoretic formulation of asynchronous exponential growth, *Trans. Amer. Math. Soc.* **303** (1987), 751–763.